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Completeness of tests of local hidden variable theories

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Abstract. Quantum mechanically for the Bohm–Aharonov system of two spin-$\frac{1}{2}$ particles in a singlet state the polarisation correlation parameter is

$$P(\hat{a},\hat{b})|_{\rm QM}=\langle\boldsymbol{\sigma}_1\,.\,\hat{a}\boldsymbol{\sigma}_2\,.\,\hat{b}\rangle=-\hat{a}\,.\,\hat{b}.$$

For a class of hidden variable theories, including those with $P(\hat{a},\hat{a})=-1$, we formulate the Einstein–Bell locality condition and obtain the following generalisation of Bell's inequality: For $N=2,3,4,\ldots,$

$$2\sum_{i<j}^{N} n_i n_j P_{ij}\leqslant\sum_{i=1}^{N} n_i^2+\tfrac{1}{2}((-1)^{\Sigma_1^N n_i}-1).$$

Here $P_{ij}\equiv P(\hat{a}_i,\hat{a}_j)$ and the n_i are integers, positive, negative or zero. For $P(\hat{a},\hat{a})=-1$, the inequalities with $n_i^2=0$ and 1 are shown to be the complete content of locality for $N=3$, 4 or 5, and not to be so for $N\geqslant 6$.

1. Introduction

Bell's theorem (Bell 1964, 1971) that no local (deterministic or stochastic) hidden variable theory can reproduce all the experimental predictions of quantum mechanics is of fundamental importance. Hence the inequalities in conflict with quantum mechanics derived by Bell (1964, 1971) and Clauser *et al* (1969) from Bell's formulation of Einstein's locality condition (Einstein 1949) have recently been the subject of extensive experimental tests (Freedman 1972, Holt 1973, Holt and Pipkin 1974, Faraci *et al* 1974, Clauser 1976, Fry and Thompson 1976, Bruno *et al* 1977, Aspect 1975, Aspect and Imbert 1976). We have shown recently that these inequalities do not exhaust the predictions of the locality condition and have proposed new experimental tests of quantum mechanics against local hidden variable theories (Roy and Singh 1978). However, the question of what constitutes a complete set of tests of local hidden variable theories has remained unanswered. The present paper answers a part of this question.

Consider the Bohm–Aharonov (Bohm and Aharonov 1957) example of a system of two spin-$\frac{1}{2}$ particles prepared in a state described quantum mechanically as a singlet, in which the two particles move in different directions. Two measuring devices measure their spin components A $(=\pm1)$ and B $(=\pm1)$ along directions $\hat{a}$ and $\hat{b}$ respectively. Then the mean value $P(\hat{a},\hat{b})$ of the product AB is given quantum mechanically by

$$P(\hat{a},\hat{b})|_{\rm QM}=\langle\boldsymbol{\sigma}_1\,.\,\hat{a}\boldsymbol{\sigma}_2\,.\,\hat{b}\rangle=-\hat{a}\,.\,\hat{b}. \tag{1}$$

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Bell (1971) characterises local hidden variable theories as those in which

$$P(\hat{a}, \hat{b}) = \int \mathrm{d}\lambda \, \rho(\lambda)\bar{A}(\hat{a}, \lambda)\bar{B}(\hat{b}, \lambda). \tag{2}$$

Here, the initial state is described by hidden variables λ with probability distribution $\rho(\lambda)$; $\bar{A}(\hat{a}, \lambda)$ and $\bar{B}(\hat{b}, \lambda)$ denote the expectation values of A and B respectively in the state λ. The chief locality assumption is that $\bar{A}(\bar{B})$ does not depend on the setting $\hat{b}(\hat{a})$ of the distant instrument. Further,

$$(\bar{A}(\hat{a}, \lambda))^2 \leqslant 1, \qquad (\bar{B}(\hat{b}, \lambda))^2 \leqslant 1 \tag{3}$$

and

$$\rho(\lambda) \geqslant 0, \qquad \int \mathrm{d}\lambda \, \rho(\lambda) = 1. \tag{4}$$

In this paper, following Bell (1964), we make the supplementary assumption

$$\bar{B}(\hat{b}, \lambda) = -\bar{A}(\hat{b}, \lambda). \tag{5}$$

The motivation for this assumption is that if the quantum mechanical result is valid at least in the case $\hat{a} = \hat{b}$, then $P(\hat{a}, \hat{a}) = -1$ requires that $\bar{B}(\hat{b}, \lambda) = -\bar{A}(\hat{b}, \lambda) = \pm 1$. Our assumption (5) is however weaker than that of Bell (1964) because we also allow $[\bar{A}(\hat{b}, \lambda)]^2 \neq 1$.

2. Inequalities

We have

$$P_{ij} \equiv P(\hat{a}_i, \hat{a}_j) = -\int \mathrm{d}\lambda \, \rho(\lambda) x_i(\lambda) x_j(\lambda); \qquad i, j = 1, 2, \ldots, N \tag{6}$$

where N is the number of settings of the measuring devices, and

$$x_i(\lambda) \equiv \bar{A}(\hat{a}_i, \lambda) \qquad (x_i(\lambda))^2 \leqslant 1. \tag{7}$$

Let $n_i (i = 1, 2, \ldots, N)$ be integers, positive, negative or zero. Then consider

$$\sum_{i<j}^{N} n_i n_j P_{ij} = -\frac{1}{2} \int \mathrm{d}\lambda \, \rho(\lambda) X(\lambda), \tag{8}$$

where

$$X(\lambda) \equiv 2 \sum_{i<j}^{N} n_i n_j x_i(\lambda) x_j(\lambda) = \Big(\sum_{i=1}^{N} n_i x_i(\lambda) \Big)^2 - \sum_{i=1}^{N} n_i^2 x_i^2(\lambda). \tag{9}$$

We seek a lower bound on $X(\lambda)$ varying the $x_i(\lambda)$ in the range -1 to $+1$. Since $X(\lambda)$ is linear in each $x_i(\lambda)$, its minimum is reached when all the $x_i(\lambda)$ lie on the boundary, i.e. $(x_i(\lambda))^2 = 1$. Hence

$$X(\lambda) \geqslant \begin{cases} 1 - \sum_{i=1}^{N} n_i^2 & \text{if } \sum_{i=1}^{N} n_i = \text{odd} \\ -\sum_{i=1}^{N} n_i^2 & \text{if } \sum_{i=1}^{N} n_i = \text{even}, \end{cases} \tag{10}$$

and from equations (4) and (8), for $N \geqslant 2$,

$$\sum_{i<j}^{N} n_i n_j P_{ij} \leqslant \frac{1}{2} \sum_{i=1}^{N} n_i^2 + \frac{1}{4}[(-1)^{\sum_{i=1}^{N} n_i} - 1]. \tag{11}$$

Here, for $\Sigma\, n_i =$ odd, we have used $(\Sigma\, n_i y_i)^2 \geqslant 1$, if $y_i = \pm 1$. Our final result is the set of inequalities (11) obtained by choosing different sets of values for the integers $n_1, n_2, \ldots, n_N$. For example, we have, for $N = 6$, $n_1 = 2$, $n_2 = n_3 = n_4 = n_5 = n_6 = 1$,

$$2(P_{12}+P_{13}+P_{14}+P_{15}+P_{16}) + (P_{23}+P_{24}+P_{25}+P_{26}+P_{34}+P_{35}+P_{36}+P_{45}+P_{46}+P_{56}) \leqslant 4 \tag{12}$$

and for any $N \geqslant 5$, $n_1 = N-4$, $n_2 = n_3 = \ldots = n_N = 1$,

$$(N-4) \sum_{j=2}^{N} P_{1j} + \sum_{1 \neq i<j}^{N} P_{ij} \leqslant \frac{1}{2}(N^2 - 7N + 14). \tag{13}$$

The special cases of (11) with $n_i = 0$ or ± 1 may be written as follows: For $M = 2, 3, \ldots, N$,

$$\sum_{i<j}^{S_M} \eta_i \eta_j P_{ij} \leqslant \begin{cases} (M-1)/2 & \text{if } M \text{ odd} \\ M/2 & \text{if } M \text{ even,} \end{cases} \tag{14}$$

where $\eta_i^2 = 1$ for $i = 1, 2, \ldots N$, $S_M \equiv (\alpha_1, \alpha_2, \ldots, \alpha_M)$ is an M-tuple of integers chosen from $1, 2, \ldots N$, and Σ^{S_M} denotes summation over index set S_M. Of these, the inequalities with even M are actually redundant. Inequalities with $M = 2$ are implied by those with $M = 3$, and inequalities with even $M = 2I$, $I \geqslant 2$, are implied by those with odd $M = 2I - I$. The inequalities with odd M are independent. For example, for $N = 5$, there are 4 inequalities for each choice of S_3, and 10 possible choices of S_3 giving 40 inequalities of $M = 3$ type; there are 16 inequalities of $M = 5$ type; thus we have 56 independent inequalities for $N = 5$. Of the 4 independent inequalities for each choice of S_3, two are equivalent to Bell's original inequalities (Bell 1964),

$$|P(\hat{a}, \hat{b}) - P(\hat{a}, \hat{c})| - P(\hat{b}, \hat{c}) \leqslant 1.$$

The inequalities (14) for odd M conflict with quantum mechanics. For $M = 3$, consider $\eta_1 \hat{a}_1$, $\eta_2 \hat{a}_2$ and $\eta_3 \hat{a}_3$ in the same plane with an angle of $2\pi/3$ between each pair of vectors, then the left-hand side of (14) is $\frac{3}{2}$ quantum mechanically, violating inequality (14). For $M = 5$, consider $\eta_4 \hat{a}_4 + \eta_5 \hat{a}_5 = 0$, and $\eta_1 \hat{a}_1$, $\eta_2 \hat{a}_2$, $\eta_3 \hat{a}_3$ in the same configuration as before; then the left-hand side of (14) equals $\frac{5}{2}$ quantum-mechanically in violation of the inequality (14). The higher M inequalities are similarly seen to conflict with quantum mechanics.

To see that the different odd M inequalities constitute independent restrictions on the P_{ij}, consider the (non quantum-mechanical) example

$$P(\hat{a}, \hat{b}) = -p_1 \hat{a} \cdot \hat{b} + p_2, \qquad p_1 \geqslant 0, \qquad p_2 \geqslant 0.$$

All the inequalities for a given odd M are satisfied if and only if

$$\tfrac{1}{2} M[p_1 + p_2(M-1)] \leqslant (M-1)/2.$$

Setting successively $M = 3, 5, 7, \ldots$ we see that each new value of M forbids new regions in the p_1, p_2 plane.

The inequalities (14) were reported earlier by Roy and Singh (1977) and independently by Selleri (1978); their special case $\eta_i = 1$ was derived from a different set of assumptions by d'Espagnat (1975). Inequalities (11) with some or all $|n_i|>1$ are believed to be new. (See e.g. equations (12), (13)). We also note that the inequalities given by us (Roy and Singh 1978) in the general case with $\bar{A}(x,\lambda)+\bar{B}(x,\lambda)\neq 0$ do also have a generalisation, corresponding to that given by (11) for the inequalities (14) discussed here.

3. Completeness

We have shown that the conditions (2)–(5) imply the inequalities (11). In particular we only used $|x_i(\lambda)|\leqslant 1$ which is a weakening of the usual condition $|x_i(\lambda)|=1$. If, however, we require $P(\hat{a},\hat{b})=-1$ for $\hat{b}=\hat{a}$, i.e. for this *one* orientation of $\hat{b}$ we demand agreement with quantum mechanics, then we must have $|x_i(\lambda)|=1$. *For the rest of this section we shall restrict ourselves to this situation.* The question of completeness is resolved by the following theorem for $N\leqslant 5$.

Theorem: Given $P_{ii}=-1$ and $P_{ij}=P_{ji}(i\neq j)$ where $i,j=1,2,\ldots,N$ for any $N\leqslant 5$ the complete content of locality is expressed by the inequalities (14).

Remarks: (i) It follows from this theorem that the inequalities obtained by allowing some or all of the $|n_i|$ to have values larger than 1 are not independent of those obtained by taking all $|n_i|\leqslant 1$.

(ii) This theorem is not true for $N>5$ in general. A counter example establishing this will be given later. Thus for $N>5$ we cannot ignore inequalities obtained by taking at least some of $|n_i|\geqslant 2$.

Proof: In order to establish the theorem we should be able to construct normalised semipositive definite $\rho(\lambda)\geqslant 0$ and $|x_i(\lambda)|=1$ provided the inequalities (14) are satisfied. We now proceed to give this explicit construction.

Divide the space of λ's into 2^{N-1} regions $D(A)$ where $A=0$ or a subset of $\{1,2,\ldots N\}$ containing no more than $N/2$ numbers, and choose

$$\xi_i,\ \xi_j x_i(\lambda)x_j(\lambda)=\chi_{i,A}\chi_{j,A}, \qquad \text{for } \lambda\in D(A),\ i<j,$$

where

$$\chi_{i,A}=\begin{cases}-1 & \text{if } i\in A\\ +1 & \text{if } i\notin A,\end{cases}$$

and ξ_i are chosen to be ± 1. $D(A)$ consists of $D(0), D(\alpha), D(\alpha,\beta), D(\alpha,\beta,\gamma),\ldots$, with the last D containing $(N-1)/2$ arguments if N is odd and $N/2$ arguments if N is even. The indices $\alpha<\beta<\ldots$ run from 1 to N. The number of different sets D with r arguments is ${}^N C_r$ except when N is even and $r=N/2$, when the number is $\frac{1}{2}{}^N C_{N/2}$. Thus

$$\chi_{i,0}=1, \qquad \chi_{i,\alpha}=(1-2\delta_{i\alpha}), \qquad \chi_{i,(\alpha,\beta)}=(1-2_{i\alpha})(1-2\delta_{i\beta}), \text{ etc.}$$

Writing

$$\rho_A=\int_{D(A)} \mathrm{d}\lambda\, \rho(\lambda)$$

and denoting $(x_i(\lambda)x_j(\lambda)\ldots x_k(\lambda))$ for $\lambda\in D(A)$ by $(x_i x_j\ldots x_k)_A$ the problem is to find

ρ_A and ξ_i satisfying

$$\sum_A \rho_A = 1,\ P_{ij} = -\sum_A \rho_A(x_i x_j)_A = -\xi_i\xi_j \sum_A \rho_A(\chi_{iA}\chi_{jA}). \tag{15}$$

N = 3: Choose $\xi_i = +1$ for $i = 1, 2, 3$. Solve for the four ρ_A's from the above four equations and obtain

$$4\rho_A = 1 - \sum_{i<j}^{3} (x_i x_j)_A P_{ij}, \qquad A = 0, 1, 2, 3. \tag{16}$$

These ρ_A's are non-negative due to the $M = 3$ inequalities in (14). The construction (16) thus obeys the locality conditions.

N = 4: Choose $\xi_i = +1$ for $i = 1$ to 4. The 7 conditions (15) do not define the $8\rho_A$'s uniquely. In terms of a free parameter P

$$-P \equiv \sum_A (x_1x_2x_3x_4)_A \rho_A, \tag{17}$$

which is a combination of the ρ_A's linearly independent from those occurring in equation (15), we find, for $A = 0, 1, 2, 3, 4, (12), (13), (23)$,

$$8\rho_A = 1 - \sum_{i<j}^{4} (x_i x_j)_A P_{ij} - (x_1x_2x_3x_4)_A P. \tag{18}$$

With the choice

$$P = - \min_{A=1,2,3,4} \left[1 - \sum_{i<j}^{4} (x_i x_j)_A P_{ij} \right], \tag{19}$$

and use of the $M = 3$ inequalities in (14), all the ρ_A are shown to be non-negative.

N = 5: Let $N_0 \equiv \max(N_3, N_5)$, where

$$(N_3, N_5) \equiv \left(\max \sum_{i<j}^{S_3} \eta_i\eta_j P_{ij},\ \frac{1}{2} \max \sum_{i<j}^{5} \eta_i\eta_j P_{ij} \right); \tag{20}$$

then the 56 inequalities (14) simply say that $N_0 \leq 1$. We choose ξ_i to equal the η_i for which the maximum N_0 is reached. Then equation (15) is equivalent to the 11 conditions

$$2 - S = 4\left[3\rho_0 + \sum_{i=1}^{5} \rho_i \right], \qquad 1 - S_{\alpha\beta} = 4(\rho_0 + \rho_\alpha + \rho_\beta + \rho_{\alpha\beta}), \tag{21}$$

where $\alpha < \beta$, α, β go from 1 to 5, and

$$S \equiv \sum_{i<j}^{5} \xi_i\xi_j P_{ij}, \qquad S_{\alpha\beta} \equiv \sum_{i<j}^{(S_3)_{\alpha\beta}} \xi_i\xi_j P_{ij}, \tag{22}$$

and $(S_3)_{\alpha\beta}$ is the triple not containing α and β (e.g. $(S_3)_{12} = (3, 4, 5)$). The general solution for the $16\rho_A$'s contains 5 free parameters P_i, $i = (1, \ldots, 5)$; with

$$P_i \equiv -\sum_A \mathrm{d}_A(x_i)_A \rho_A, \qquad \mathrm{d}_A(x_i)_A \equiv (x_1x_2x_3x_4x_5)_A (x_i)_A,$$

$$16\rho_A = 1 - \sum_{i<j}^{5} (x_i x_j)_A P_{ij} - \sum_{i=1}^{5} \mathrm{d}_A(x_i)_A P_i.$$

If $N_0 = N_5$ we may choose

$$16\rho_A = 1 - N_0, \qquad A = 0, 1, \ldots 5; \qquad 16\rho_{\alpha\beta} = (1 - N_0) + 4(N_0 - S_{\alpha\beta}), \qquad \alpha < \beta, \quad (23)$$

which satisfy the condition (21). The ρ_A's are obviously non-negative.

If $N_0 = N_3$, we may assume without loss of generality that $N_0 = S_{45}$. We may choose

$$\begin{aligned}
&\rho_0 = \rho_4 = \rho_5 = (1 - N_0)/16 \\
&16\rho_1 = (1 - N_0) + 4\max[(N_0 - S + S_{23})_+, 2N_0 - S - C_2 - C_3] \\
&16\rho_3 = (1 - N_0) + 4\min[2N_0 - S - (N_0 - S + S_{23})_+ - (N_0 - S + S_{13})_+, C_3] \\
&16\rho_2 = 3(1 - N_0) + 4(2N_0 - S) - 16(\rho_1 + \rho_3) \\
&4\rho_{\alpha\beta} = (1 - S_{\alpha\beta}) - 4(\rho_0 + \rho_\alpha + \rho_\beta) \\
&\alpha < \beta, \alpha, \beta \in (1, 2, 3, 4, 5)
\end{aligned} \qquad (24)$$

where we denote $X_+ = \max(0, X)$ and

$$C_i \equiv \min(N_0 - S_{i4}, N_0 - S_{i5}), \qquad i = 2, 3. \qquad (25)$$

It is straightforward to verify that the above ρ_A's are non-negative and satisfy the conditions (21).

We have thus completed the proof for $N \leqslant 5$. One may be naturally tempted at this stage to conjecture that the restriction of inequalities (11) to $n_i = 0, \pm 1$ may be sufficient to express the content of locality for all N. That this is not so is brought out by the following counter example for $N = 6$. Let P_{ij} be equal to $P_{ij}^{(0)}$ where

$$P_{ij}^{(0)} = -\delta_{ij} + \tfrac{1}{2}(1 - \delta_{ij})(\delta_{i1} + \delta_{j1}) \qquad i, j = 1, 2, \ldots, 6. \qquad (26)$$

This set of P_{ij}'s satisfies all the inequalities (11) with $n_i = 0, \pm 1$ but the matrix $\|P_{ij}^{(0)}\|$ is not a negative semidefinite matrix and as such has no representation of the form (6). This set of P_{ij}'s however does not satisfy the inequalities (11) with no restriction of n_i's. For example

$$\sum_{i<j} n_i n_j P_{ij}^{(0)} = \frac{1}{2} n_1(n_2 + n_3 + \ldots + n_6) \leqslant \frac{1}{2}\sum_{i=1}^{6} n_i^2 + \frac{1}{4}[(-)^{n_1 + \ldots + n_6} - 1]$$

is incorrect for $n_1 = 2$, $n_2 = n_3 = n_4 = n_5 = n_6 = 1$.

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